Equation of motion for interacting pulses

Shin-Ichiro Ei[1] and Takao Ohta[2]
[1]Graduate School of Integrated Science, Yokohama City University, Yokohama 236, Japan
[2]Department of Physics, Ochanomizu University, Tokyo 112, Japan
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We develop a systematic method of deriving the equation of motion for interacting fronts or pulses in one dimension. The theory is applicable to both dissipative and dispersive systems. In the case of the time-dependent Ginzburg-Landau equation, which is a typical example of a dissipative system, the front equation obtained is the same as has been obtained previously. The pulse interaction is also derived for the Korteweg–de Vries equation, emphasizing the difference between the cases with and without dissipative terms.

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I. INTRODUCTION

In this paper, we shall study dynamics of interacting pulses in one dimension. A systematic theory is developed to derive the equation of motion for a pair of pulses. Our method does not rely on the existence of Lyapunov or free energy functional of the time-evolution equation. Furthermore, the theory can be applied equally to both dissipative and dispersive systems. The basic assumption is that the distance between two pulses is much larger than the pulse width.

In order to illustrate our method, we consider two representative systems. One is the kink-antikink interaction in a time-dependent Ginzburg-Landau (TDGL) equation which is familiar in the theory of phase transitions. The TDGL equation in one dimension takes the following form for the field variable u:

$$\frac{\partial u}{\partial t}=\varepsilon^2\frac{\partial^2 u}{\partial x^2}+f(u)\ , \tag{1.1}$$

where ε is a positive constant. The nonlinear function $f(u)$ is given by

$$f(u)=\tfrac{1}{2}u(1-u^2)\ . \tag{1.2}$$

Note that Eq. (1.1) can be written as

$$\frac{\partial u}{\partial t}=-\frac{\delta F}{\delta u}\ , \tag{1.3}$$

where

$$F\{u\}=\int dx\left[\frac{\varepsilon^2}{2}\left[\frac{\partial u}{\partial x}\right]^2+W(u)\right]\ , \tag{1.4}$$

with $dW/du=-f$. Thus, the TDGL equation (1.1) is a typical example of a dissipative system with a free energy (Lyapunov) functional $F\{u\}$.

The TDGL equation has a nonuniform equilibrium solution,

$$u(x,t)=\pm U(x)=\pm\tanh(x/2\varepsilon)\ , \tag{1.5a}$$

where $U(x)$ satisfies

$$\varepsilon^2\frac{\partial^2 U}{\partial x^2}+f(U)=0\ . \tag{1.5b}$$

We call the solution with the plus (minus) sign a kink (an antikink). When there is a pair of kink and antikink in an infinite extended system, the kink and antikink have to move because such a configuration of u is not a stationary solution of (1.1). This problem was addressed by Kawasaki and Ohta [1]. They derived the interaction of a kink-antikink pair assuming that the kink width ε is sufficiently small compared to the distance between the kink and antikink positions. The interaction was found to be attractive. Later, Carr and Pego [2] also considered the same problem relying fully on the existence of the free energy functional $F\{u\}$. Their result is consistent with that of Kawasaki and Ohta except for a minor difference of a factor of two in the interaction strength [3].

The other example is the Kortweg–de Vries (KdV) equation with dissipative terms:

$$\frac{\partial u}{\partial t}+6u\frac{\partial u}{\partial x}+\frac{\partial^3 u}{\partial x^3}+a\left[\frac{\partial^2 u}{\partial x^2}+\frac{\partial^4 u}{\partial x^4}\right]=0\ , \tag{1.6}$$

where a is a positive constant. Equation (1.6) was first derived and studied by Benney [4] and is called sometimes Benney equation. As is well known, Eq. (1.6) is completely integrable when the dissipative terms are absent, i.e., $a=0$. The KdV equation admits propagating pulse (soliton) solutions. Because of the integrability, the collision of a pair of pluses can be analyzed in a rigorous manner. However, a completely integrable system is quite exceptional in nature. Some dissipation such as (1.6) is not avoidable in any realistic systems. Therefore, to develop a systematic method of deriving the pulse interaction is necessary for a perturbed KdV equation which is not integrable any more.

There are several previous results for the pulse interaction of KdV equation and its modified version. The interaction for a dispersive KdV equation with a fifth derivative term was performed in Refs. [5] and [6]. Kawahara and Takaoka [7] derived the equation of motion for pulses of (1.6) with $a \neq 0$. However, it seems to us that the validity of their results is questionable since the equation of motion obtained has a time-reversal symmetry despite the fact that the starting equation (1.6) does not.

The pulse interaction in a dissipative system with no Lyapunov functional has been studied by Yamada and Nozaki [8]. They have considered the FizHugh-Nagumo equation which is a model equation for pulse propagation along a nerve axon [9]. Although their method is close somehow to ours, they have not applied it to dispersive systems.

As was mentioned in the beginning of this section, the purpose of this paper is to develop a systematic theory of interaction pulses, which is capable of dealing with both dissipative and dispersive systems in a unified way. We emphasize that arbitrary parameters contained in a pulse solution play a central role in the coarse-grained description of pulse dynamics. A simple but nontrivial example in a continuum system is the position of a localized solution. Its specification violates the translational symmetry of the system and hence the position is a kind of Goldstone mode. Thus, when we consider weak deformations of a localized solution, the position is a relevant slow variable. This is the basic idea in the theory of interface and/or phase dynamics [10].

However, one often encounters the situation where there are several free parameters. The pure KdV equation is a typical example since the velocity of a propagating pulse as well as its position is arbitrary. Thus, one needs to take into account the extra degrees of freedom associated with the velocity deviation to formulate the pulse dynamics. What is remarkable in this problem is that when a weak dissipation is present as in (1.6) the velocity is uniquely determined asymptotically. Thus, one can see how the pulse dynamics is altered by this velocity selection. This is another reason as to why we are concerned with KdV equation in the weak dissipation limit.

In Sec. II we derive the interaction of kinks in the TDGL equation (1.1) while in Sec. III we consider the pulse interaction in the pure KdV equation. The effects of the dissipative terms in the limit $a \to 0$ will be studied separately in Sec. IV. The final section (Sec. V) is devoted to discussions. Some technical details of the solvability condition used in Sec. III are described in Appendix.

II. KINK-ANTIKINK INTERACTION IN TDGL EQUATION

First, we consider the kink-antikink interaction in the TDGL equation. Although the results in this section are not essentially new, the many formulas obtained here will be useful in the study of the pulse interaction in the later sections.

Suppose that there are a kink at $x = x_1$ and an antikink at $x = x_2$ $(> x_1)$. We assume that $x_2 - x_1 \gg \varepsilon$, where ε is the width of kink and antikink. In this situation, we may put the solution $u(x,t)$ of (1.1) as

$$u(x,t) = \tilde{U}(x) + b(x,t) , \tag{2.1}$$

where

$$\tilde{U}(x) = U(x - x_1) - U(x - x_2) - 1 . \tag{2.2}$$

The function $\tilde{U}(x)$ is a superposition of the kink-antikink pair and $b(x,t)$ stands for a deviation which is expected to be small when $x_2 - x_1 \gg \varepsilon$. Our aim is to derive the time-evolution equation for the positions x_1 and x_2.

As was discussed in Ref. [1] and at the end of this section, the small parameter in the present problem is $\exp[-(x_2 - x_1)/2\varepsilon]$. As a result, one will see that the motion of interacting kinks is very slow and is of the order of $\exp[-(x_2 - x_1)/\varepsilon]$. However, we do not introduce a scaled time in the theory shown below because no confusion is expected to occur.

Substituting (2.1) into (1.1) yields up to order b,

$$-\dot{x}_1 U_1' + \dot{x}_2 U_2' + b_t = Lb + f(\tilde{U}) - f(U_1) + f(U_2) , \tag{2.3}$$

where the dot means the time derivative and the prime the derivative with respect to the argument. We have used abbreviation of notations such that $U_i = U(x - x_i)$. Note that U_1 and U_2 satisfy Eq. (1.5b). Hence, Eq. (2.3) is exact up to $O(b)$. The linear operator L is defined by

$$L = \varepsilon^2 \frac{d^2}{dx^2} + f'(\tilde{U}) . \tag{2.4}$$

Hereafter, we focus our attention on the motion of the kink position x_1. In the vicinity of $x = x_1$, the tail of the antikink $1 + U_2$ is negligibly small so that one may use the following expansion:

$$f(\tilde{U}) - f(U_1) + f(U_2) = -3\varepsilon(1 + U_2)U_1' + \tfrac{3}{2}(1 + U_2)^2(1 - U_1) , \tag{2.5}$$

where the relation $1 - U_1^2 = 2\varepsilon U_1'$ has been used. Because $f(u)$ is a cubic function as Eq. (1.2) the expansion (2.5) is rigorous. Note that a term of the order of $(1 + U_2)^3$ is absent in (2.5). Similarly, one can replace the operator L by L_{GL} defined by

$$L_{GL} = \varepsilon^2 \frac{d^2}{dx^2} + f'(U_1) . \tag{2.6}$$

Since L_{GL} is the operator which appears in linearization of (1.1) around a kink solution, it is obvious that it has zero eigenvalue $\lambda_0 = 0$ with the (unnormalized) eigenfunction $\Phi = U_1'$. This is simply the translation mode of a kink. The eigenvalue problem for L_{GL} can be solved exactly. See, for instance, Ref. [11]. There are two discrete states with the eigenvalues $\lambda_0 = 0$ and $\lambda_1 = -\frac{3}{4}$. The eigenvalue of the continuous state is given by $\lambda_p = -(1 + p^2)$ $(-\infty < p < \infty)$.

The equation of motion for a kink position can be obtained from the solvability condition of Eq. (2.3) such that the inhomogeneous terms should be orthogonal to the zero eigenfunction of the self-adjoint operator L_{GL}. Thus, we obtain

$$\dot{x}_1(U_1',U_1')=-(U_1',f(\tilde{U})-f(U_1)+f(U_2)) , \quad (2.7)$$

where $(A,B)=\int_{-\infty}^{\infty}dx\, A(x)B(x)$. The equation for x_2 can also be obtained similarly. The result is the same form as (2.7) with the replacements $\dot{x}_1$ by $-\dot{x}_2$ and U_1' by U_2'.

Now we evaluate the integrals in (2.7) to obtain the equation of motion explicitly. The coefficient (U_1',U_1') is readily calculated as

$$(U_1',U_1')=\frac{2}{3\varepsilon} , \quad (2.8)$$

where we have used the formula for $|a|<4$,

$$\int_{-\infty}^{\infty}dx\frac{\exp(ax)}{\cosh^4 x}=\frac{4}{3}\left[1-\left[\frac{a}{2}\right]^2\right]\frac{\pi a/2}{\sin(\pi a/2)} . \quad (2.9)$$

In order to evaluate the right hand side of (2.7), one may use the expansion (2.5):

$$-(U_1',f(\tilde{U})-f(U_1)+f(U_2))$$
$$=3\varepsilon(U_1',(1+U_2)U_1')-(U_1',\tfrac{3}{2}(1+U_2)^2(1-U_1)) . \quad (2.10)$$

The first term in (2.10) can be written as

$$3\varepsilon(U_1',(1+U_2)U_1')=\tfrac{3}{2}\int_{-\infty}^{\infty}dz\frac{1}{\cosh^4 z}\left[1+\tanh\left[z-\frac{x_2-x_1}{2\varepsilon}\right]\right]$$
$$\approx 3\int_{-\infty}^{L}dz\frac{1}{\cosh^4 z}\sum_{n=1}^{\infty}(-1)^{n+1}\exp(2nz)\exp\left[-2n\frac{x_2-x_1}{2\varepsilon}\right]+3\int_{L}^{\infty}dz\frac{1}{\cosh^4 z} , \quad (2.11)$$

where $L=(x_2-x_1)/2\varepsilon$ and we have used the expansion for $z<0$

$$1+\tanh z=2\sum_{n=1}^{\infty}(-1)^{n+1}\exp(2nz) . \quad (2.12)$$

Up to order of $\exp[-(x_2-x_1)/\varepsilon]$, one may retain only the $n=1$ term in (2.11) and extend the integral domain from $-\infty$ to ∞. Using the formula (2.9) again, we obtain

$$\dot{x}_1=12\varepsilon\exp[-(x_2-x_1)/\varepsilon)] . \quad (2.13)$$

This is consistent with that obtained by Carr and Pego [2].

Kawasaki and Ohta [1] obtained the same result but with the coefficient 6 instead of 12. This discrepancy originates from the fact that they employed the following approximation:

$$1+U_2\approx 2\exp\left[\frac{x_1-x_2}{\varepsilon}\right] . \quad (2.14)$$

However, the proper treatment up to order $\exp[-(x_2-x_1)/\varepsilon]$ is to use (2.11) as shown above.

The equation for the antikink position x_2 is the same form as (2.13), except for the minus sign in front of the right hand side. Recall that we have put $x_2>x_1$. Thus, we note that x_1 increases with time whereas x_2 decreases. This means that the kink-antikink interaction is attractive although the strength is very small for $(x_2-x_1)/\varepsilon\gg 1$ as we have assumed. A kink-antikink pair annihilate upon collision in this dissipative system.

We make a remark that our method presented here is not limited to the lowest order. (2.11) shows that the next order correction is of the order of $[(x_2-x_1)/\varepsilon]\exp[-2(x_2-x_1)/\varepsilon]$. The corrections arising from the second term in (2.10) can also be evaluated similarly which is of the order of $\exp[-3(x_2-x_1)/2\varepsilon]$. These indicate that the corrections can be obtained essentially by the expansion in terms of $\exp[-(x_2-x_1)/2\varepsilon]$. Furthermore, the eigenvalue problem associated with (2.4) is solved by a perturbation expansion. Thus, the higher order corrections in the kink interaction can, in principle, be calculated without any difficulty.

III. PULSE INTERACTION IN KdV EQUATION

In this section, we derive the pulse interaction of Eq. (1.6) without the dissipative terms, i.e., $a=0$. In this case, one pulse solution is well known and is given by

$$u(x,t)=V(c,x-ct)=\frac{c}{2}\mathrm{sech}^2\left[\frac{\sqrt{c}}{2}(x-ct)\right] , \quad (3.1a)$$

where $V(c,x-ct)$ satisfies

$$-c\frac{\partial V}{\partial x}+6V\frac{\partial}{\partial x}V+\frac{\partial^3 V}{\partial x^3}=0 . \quad (3.1b)$$

It should be noted that the velocity of the propagating pulse denoted by c is not specified but an arbitrary positive constant.

We consider the interaction of two pulses located at $x=x_1(t)$ and $x_2(t)$ propagating with almost identical velocities. The distance x_2-x_1 is assumed to be much larger than the pulse width $1/\sqrt{c}$. The solution $u(x,t)$ can be written as

$$u(x,t)=V(c+\dot{x}_1,x-ct-x_1)$$
$$+V(c+\dot{x}_2,x-ct-x_2)+b(x-ct,t) . \quad (3.2)$$

We assume that $\dot{x}_1$ and $\dot{x}_2$ which arise from the interaction are sufficiently small compared to c. This will be checked self-consistently in the final results. Substituting (3.2) into (1.6) with $a=0$, one obtains up to $O(b)$,

$$b_t=Mb+g , \quad (3.3a)$$

where

$$g=-\sum_{i=1,2}\left[\ddot{x}_i\frac{\partial V_i}{\partial c}-(c+\dot{x}_i)\frac{\partial V_i}{\partial x}+\frac{\partial^3 V_i}{\partial x^3}\right]$$
$$-6(V_1+V_2)\frac{\partial}{\partial x}(V_1+V_2) \tag{3.3b}$$

and $V_i=V(c+\dot{x}_i,x-ct-x_i)$ $(i=1,2)$. The operator M is given by

$$M=c\frac{\partial}{\partial x}-\frac{\partial^3}{\partial x^3}-6\sum_{i=1,2}\left[\frac{\partial V_i}{\partial x}+V_i\frac{\partial}{\partial x}\right] . \tag{3.4a}$$

Now we examine the property of the operator M. In the vicinity of the position $x=x_1$, the pulse solution V_2 is sufficiently small so that one may ignore V_2 in (3.4a). Thus, the operator M is simplified as

$$M=c\frac{\partial}{\partial x}-\frac{\partial^3}{\partial x^3}-6\left[\frac{\partial V}{\partial x}+V\frac{\partial}{\partial x}\right] . \tag{3.4b}$$

Here, and in what follows, we occasionally omit the suffix 1 in V_1 when no confusion arises. One of the zero eigenfunctions of M is given, as in the case of the TDGL equation, by

$$\Phi_1=\frac{\partial V}{\partial x} . \tag{3.5a}$$

It is emphasized, however, that there is another zero eigenfunction for M, which is given by

$$\Phi_2=\frac{\partial V}{\partial c} . \tag{3.5b}$$

In fact, it is readily shown that

$$M\Phi_2=-\Phi_1 \tag{3.6a}$$

and hence

$$M^2\Phi_2=-M\Phi_1=0 . \tag{3.6b}$$

The relation (3.6a) cam be obtained from (3.1b) by differentiating it with respect to c. This property is a consequence of the fact that the speed c in the pulse solution (3.1) is arbitrary in the KdV equation. The degeneracy of the zero eigenstate requires caution in applying the solvability condition for (3.3) as will be shown below and in Appendix.

We need to introduce the adjoint operator $M^\dagger$ of M:

$$M^\dagger=-c\frac{\partial}{\partial x}+\frac{\partial^3}{\partial x^3}+6(V_1+V_2)\frac{\partial}{\partial x}$$
$$\approx -c\frac{\partial}{\partial x}+\frac{\partial^3}{\partial x^3}+6V_1\frac{\partial}{\partial x} . \tag{3.7a}$$

Thus, we have essentially the same eigenvalue problem as in the previous section:

$$\left[-c+6V_1+\frac{d^2}{dx^2}\right]\psi=\lambda\psi . \tag{3.7b}$$

In this case, however, there are three discrete and one continuous states [11]. The eigenvalues of the discrete states are given by $\lambda_{-1}=5c/4$, $\lambda_0=0$, and $\lambda_1=-3c/4$. The eigenfunction ψ for the zero mode is

$$\psi=-\operatorname{sech}^2\left[\frac{\sqrt{c}}{2}z\right]\tanh\left[\frac{\sqrt{c}}{2}z\right] , \tag{3.8}$$

where $z=x-x_1$. The zero eigenfunction Ψ of $M^\dagger$ is, therefore, obtained by integrating ψ under the condition $\Psi\to 0$ for $z\to\pm\infty$),

$$\Psi=\operatorname{sech}^2\left[\frac{\sqrt{c}}{2}z\right] . \tag{3.9a}$$

Without loss of generality, one may equate it with V,

$$\Psi(x)=V(c,z) . \tag{3.9b}$$

Note that there is no other localized zero mode. It should be mentioned here that only the zero eigenfunction ψ is related with Ψ as $d\Psi/dz=\psi$. Other eigenfunctions do not have such a simple relation.

Now we derive the equation for x_1. Since the operator M is not self-adjoint and has degenerate zero states, it is not *a priori* obvious whether or not the orthogonality condition for Ψ and the inhomogeneous term in (3.3a) is the proper solvability condition. What one should require is that the solution b in (3.3a) must be bounded for $t\to\infty$. In the Appendix, we prove that this is indeed equivalent with the condition:

$$(g,\Psi)=0 . \tag{3.10a}$$

This leads us to the equation for x_1,

$$\ddot{x}_1\left[\frac{\partial V_1}{\partial c},V_1\right]+6\left[\frac{\partial}{\partial x}V_1V_2,V_1\right]=0 , \tag{3.10b}$$

where we have used (3.1b) and the facts that $(\partial V_1/\partial x,V_1)=0$ and $(\partial V_2/\partial c,V_1]\approx 0$ for $(x_2-x_1)/\sqrt{c}\gg 1$. Since

$$\frac{\partial V_2}{\partial c}=\frac{1}{c}V_2+\frac{z}{2c}\frac{\partial V_2}{\partial z} , \tag{3.11}$$

one obtains

$$\left[\frac{\partial V_1}{\partial c},V_1\right]=\frac{3}{4c}(V_1,V_1)=\frac{\sqrt{c}}{2} , \tag{3.12}$$

where (2.9) with $a=0$ has been used. On the other hand, the second term in (3.10b) can be obtained as

$$6\left[\frac{\partial}{\partial x}V_1V_2,V_1\right]=3\left[\frac{\partial}{\partial x}V_2,V_1^2\right]$$
$$\approx 8c^3\exp[-\sqrt{c}(x_2-x_1)] . \tag{3.13}$$

In this derivation, we have used the asymptotic form $V_2\approx 2c\exp[\sqrt{c}(z-x_2)]$ for $z\ll x_2$ and the formula (2.9) with $a=2$. From (3.10b), (3.12), and (3.13) one obtains up to order of $\exp[-\sqrt{c}(x_2-x_1)]$,

$$\ddot{x}_1=-16c^{5/2}\exp[-\sqrt{c}(x_2-x_1)] . \tag{3.14}$$

The equation for x_2 is given by changing $\ddot{x}_1$ by $-\ddot{x}_2$. Thus, the pulse interaction is found to be repulsive in this case. As in the case of the TDGL equation in the previous section, the higher order corrections to (3.14) can be obtained by the expansion in powers of $\exp[-\sqrt{c}(x_2-x_1)]$.

The distance $l(t)=x_2-x_1$ obeys

$$\ddot{l}=32c^{5/2}\exp[-\sqrt{c}\,l(t)] . \tag{3.15}$$

One can study the collision of two pulses by solving Eq. (3.15). Since the interaction is repulsive, there must be a minimum distance $l_{\min}$ (turning point) where two pulses change their propagating direction in the moving frame with the velocity c. From (3.15), one obtains

$$l_{\min}=\frac{2}{\sqrt{c}}\ln\frac{8c}{\Delta c} , \tag{3.16}$$

where Δc is the relative velocity at $t=-\infty$. The position shift after collision can be obtained rigorously from the two-pulse solution [12]. If one equates it with $l_{\min}$ one notes that the rigorous result gives us (3.16) with the factor $4c$ instead of $8c$. This is not surprising because Eq. (3.15) is an asymptotic result correct for $(x_2-x_1)\gg 1/\sqrt{c}$.

The result (3.14) can be generalized to an n-pulse system. Then the equation of motion for the positions x_i $(i=1,\ldots,n)$ is found to be given by the Toda lattice equation [5,6] which is also completely integrable. This is an important feature of our method in the sense that the complete integrability of the starting KdV equation is preserved under the reductive representation of the pulse interaction in terms of $\{x_i\}$.

IV. PULSE INTERACTION IN BENNEY EQUATION

When the KdV equation has a dissipative term as (1.6), the interaction among pulses is expected to be modified qualitatively. Here, we explore this problem in two steps. First, when the dissipative terms are present, the speed of the propagating pulse is shown not to be arbitrary but is determined uniquely. Next, the pulse interaction is obtained by generalizing the method in the previous section. Throughout this section, we assume that the parameter a in (1.6), which is a measure of the strength of the dissipative terms, is sufficiently small, $0<a\ll 1$.

When a is small, the effect of dissipation is expected to appear in the time scale $1/a$ so that we introduce a scaled time $T=at$. First, let us consider a single pulse and determine the propagating velocity. A similar problem has been considered in Ref. [15]. We expand $u(x,t)$ in powers of a

$$u(x,t)=u^{(0)}(x,t,T)+au^{(1)}(x,t,T)+\cdots \tag{4.1}$$

and substitute it into (1.6). The zeroth order solution is in the same form as $V(x-ct-X_0,c)$ given by (3.1) except for the fact that the parameters c and X_0 may depend on the scaled time T. From the first order terms, one obtains for $u^{(1)}=u^{(1)}(x-ct-X_0,t)$

$$\frac{\partial u^{(1)}}{\partial t}=Mu^{(1)}+G , \tag{4.2}$$

where the operator M is given by (3.4b). The inhomogeneous term G takes the following form

$$G=At+B , \tag{4.3a}$$

where

$$A=\frac{dc}{dT}\frac{\partial V}{\partial z} , \tag{4.3b}$$

$$B=-\frac{dX_0}{dT}\frac{\partial V}{\partial z}-\frac{dc}{dT}\frac{\partial V}{\partial c}-\frac{\partial^2 V}{\partial z^2}-\frac{\partial^4 V}{\partial z^4} , \tag{4.3c}$$

with $z=x-ct-X_0$.

We need to impose the condition that the secular term in the solution of Eq. (4.2) should vanish. This can be achieved in the manner similar to that in the Appendix. Using the projection operator P defined by (A2), the secular part can be written as

$$Pu^{(1)}(t)=(I+tM_1)Pu^{(1)}(0)+\int_0^t ds\{I+(t-s)M_1\}PG(s) , \tag{4.4a}$$

where I is a unit operator and we have used the relation (A3). Substituting (4.3b) and (4.3c) into (4.4a), one obtains

$$Pu^{(1)}(t)=Pu^{(1)}(0)+t[MPu^{(1)}(0)+PB]+\tfrac{1}{2}t^2(PA+PMB) , \tag{4.4b}$$

where we have ignored the slow time dependence of G through T. Thus, the required condition is given by

$$PA+PMB=0 . \tag{4.5}$$

Substituting (4.2b), (4.2c), and (A1) with $\varphi=B$ and noting the relations (3.5a) and (3.6a), one finally obtains from (4.5)

$$\frac{dc}{dT}=\frac{1}{2}\frac{\left[\frac{\partial^2 V}{\partial z^2}+\frac{\partial^4 V}{\partial z^4},\Psi\right]}{\left[\frac{\partial V}{\partial z},\tilde{\Psi}\right]} , \tag{4.6a}$$

where Ψ has been given by (3.9b) and $\tilde{\Psi}$ is defined through the relation (A2) in the Appendix. It is readily shown that (4.6a) is consistent with that obtained by a direct expansion [13]. The asymptotic speed of the pulse, which is denoted by $c=c^*$, is given by the condition

$$\left[\frac{\partial^2 V}{\partial z^2}+\frac{\partial^4 V}{\partial z^4},\Psi\right]=0 . \tag{4.6b}$$

The formulas (4.6) are correct up to the lowest order of a.

Now we consider the pulse interaction. Suppose that there is a single pulse which obeys the pure KdV equation with $a=0$. We switch on the dissipative terms at some instant. The pulse profile as well as its velocity changes gradually to the asymptotic form with $c=c^*$. We are concerned with the interaction between these asymptotic pulses. We put two asymptotic pulses at $x=x_1$ and x_2 and see how these pulses interact with each other. Since the speed $c(a)$ satisfying $c(0)=c^*$ is uniquely determined, we may write the solution $u(x,t)$ of Eq. (1.6) as

$$u(x,t)=V_a(z-x_1)+V_a(z-x_2)+b(z,t) , \tag{4.7}$$

with $z=x-c(a)t$. The one-pulse solution is denoted by V_a emphasizing the finiteness of a. It satisfies

$$\left[-c(a)\frac{d}{dz}+\frac{d^3}{dz^3}+6V_a\frac{d}{dz}+a\left[\frac{d^2}{dz^2}+\frac{d^4}{dz^4}\right]\right]V_a=0\ . \tag{4.8}$$

By the same method as in Sec. III, we obtain the equation for x_1,

$$\gamma\dot{x}_1+6\left[\frac{\partial}{\partial z}(V_{a1}V_{a2}),\Psi_a\right]=0\ , \tag{4.9}$$

where

$$\gamma=-\left[\Psi_a,\frac{dV_a}{dz}\right]\ . \tag{4.10}$$

The function Ψ_a obeys the equation similar to (4.8),

$$\left[-c(a)\frac{d}{dz}+\frac{d^3}{dz^3}+6V_a\frac{d}{dz}-a\left[\frac{d^2}{dz^2}+\frac{d^4}{dz^4}\right]\right]\Psi_a=0\ . \tag{4.11}$$

It should be noted that the $\ddot{x}$ term does not exist in (4.9) since the velocity has been fixed to be the terminal one $c=c(a)$.

In the limit $a\to 0$, the coefficient γ vanishes identically as shown in Sec. III. Here we evaluate it by the expansion in terms of a. Up to order a, V_a, Ψ_a, and $c(a)$ can be written as

$$V_a(z)=V(z)+aV^{(1)}(z)\ , \tag{4.12a}$$

$$\Psi_a(z)=\Psi(z)+a\Psi^{(1)}(z)\ , \tag{4.12b}$$

$$c(a)=c^*+ac^{(1)}\ . \tag{4.12c}$$

The first order correction of the velocity has been studied recently [14]. The lowest order solutions V and Ψ are given, respectively, by (3.1a) and (3.9b) with $c=c^*$. From(4.8) one notes that the first order correction $V^{(1)}$ satisfies

$$MV^{(1)}=\frac{\partial^2 V}{\partial z^2}+\frac{\partial^4 V}{\partial z^4}-c^{(1)}\frac{\partial V}{\partial z}\ . \tag{4.13}$$

The operator M is the same as (3.4b) with $c=c^*$. The other correction $\Psi^{(1)}$ obeys a slightly different equation:

$$M^\dagger\Psi^{(1)}=-6V^{(1)}\frac{d\Psi}{dz}+\frac{\partial^2 V}{\partial z^2}+\frac{\partial^4 V}{\partial z^4}+c^{(1)}\frac{d\Psi}{dz}\ , \tag{4.14}$$

with $M^\dagger$ defined by (3.7a).

The coefficient γ can be expanded up to order a as

$$\gamma=-a\left[\frac{\partial V}{\partial z},\Psi^{(1)}\right]-a\left[\frac{\partial V^{(1)}}{\partial z},\Psi\right]\ . \tag{4.15}$$

In order to calculate the first term in (4.15), one notes the relation

$$\left[\frac{\partial V}{\partial z},\Psi^{(1)}\right]=-\left[\frac{\partial V}{\partial c},M^\dagger\Psi^{(1)}\right]\ . \tag{4.16}$$

In this derivation, we have used (3.6a). From (4.13) and (4.14), one obtains after putting $\Psi=V$ as (3.9b),

$$M^\dagger\Psi^{(1)}=M^\dagger V^{(1)}+2\left[\frac{\partial^2 V}{\partial z^2}+\frac{\partial^4 V}{\partial z^4}\right]\ . \tag{4.17}$$

Putting (4.15)–(4.17) together and using (3.6a) again, the coefficient γ is given up to $O(a)$ by

$$\gamma=2a\left[\frac{\partial V}{\partial c},\frac{\partial^2 V}{\partial z^2}+\frac{\partial^4 V}{\partial z^4}\right]\ . \tag{4.18}$$

Note that a first order correction $c^{(1)}$ does not appear in (4.18). One can simplify (4.18) further. Noting the facts that (3.11) and (4.6b) with $\Psi=V$, one finally obtains

$$\gamma=\frac{a}{c^*}\left[\frac{\partial V}{\partial z},\frac{\partial V}{\partial z}\right]+O(a^2)\ . \tag{4.19}$$

This clearly indicates that γ is positive. (4.19) is readily evaluated so that Eq. (4.9) can be written as

$$\dot{x}_1=-\frac{120}{a}c^{*3/2}\exp[-\sqrt{c^*}(x_2-x_1)]\ . \tag{4.20}$$

The result (4.20) indicates that the interaction between two pulses turns out to be repulsive under the dissipation. This implies that in a system having many pulses the distance between adjacent pulses tend to be equal asymptotically due to the interaction.

V. DISCUSSIONS

We have developed a systematic method dealing with the pulse dynamics in both dissipative and dispersive systems. The method does not rely on any Lyapunov functional so that it can be applied to systems far from equilibrium. The smallness parameter of the problem takes the form of $\exp(-L/b)$, where L is the pulse distance and b the pulse width.

We have shown in Sec. III that any free parameters in a one-pulse solution play an important role in the pulse interaction. In the case of KdV equation, the fact that the velocity is a free parameter guarantees that the pulse equation of motion has a time-reversal symmetry. If weak dissipation is present, the velocity is fixed and a friction term appears in the asymptotic pulse equation of motion. The velocity selection of a propagating pulse in KdV equation is also formulated in the presence of small dissipations.

Dynamics of pulse and front has often been studied by a perturbation expansion in terms of a solvable limit. A typical example is a complex Ginzburg-Landau equation [15], where one may start either from a dissipative limit as (1.3) or from a Hamiltonian limit like a nonlinear Schrödinger equation. Perturbations generally break some symmetry or conservation laws which exist in the limiting systems. This makes the free parameters in the lowest solution fix. These aspects are similar to those studied in Sec. III.

We have emphasized that the $a\to 0$ limit in Eq. (1.6) is a singular limit in a sense that the propagating velocity is uniquely determined for $a\neq 0$, whereas it is arbitrary for $a=0$. Thus the property of the pulse interaction is qualitatively different in these two cases. In the presence of dissipations, the second derivative of the pulse position with respect to time (the inertia term) should not exist in the asymptotic equation of motion. This differs from the previous theory where the inertia term is present even for strong dissipations. In an intermediate time regime, both inertia and friction terms are expected to exist. In fact, Elphick, Regev, and Spiegel [16] have derived such a pulse equation of motion from Eq. (1.6). However, their expression looks apparently quite different from ours Eq. (4.20) because they do not assume the smallness of a. Since the a dependence of the coefficients is not given explicitly in their results, a direct comparison with our theory is not possible at present.

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APPENDIX

In this Appendix, we prove that the absence of the secular term in the solution of (3.3) is equivalent with the orthogonality condition (3.10). We confine ourselves to the vicinity of the pulse position $x=x_1$. Since the zero state of the operator M is degenerate and has the eigenfunctions Φ_1 and Φ_2 as given by (3.5), one needs to introduce a projection operator P defined by

$$P\varphi=\frac{(\varphi,\tilde{\Psi})}{(\Phi_1,\tilde{\Psi})}\Phi_1+\frac{(\varphi,\Psi)}{(\Phi_2,\Psi)}\Phi_2 , \tag{A1}$$

where φ stands for the eigenspace of M. We have introduced $\tilde{\Psi}$ such that

$$M^{\dagger}\tilde{\Psi}=\Psi , \tag{A2}$$

where Ψ is the zero eigenstate of $M^{\dagger}$ given by (3.9). It should be noted that

$$M^2P=0 . \tag{A3}$$

Using the operator P, the solution of (3.3), which would contain the secular part is given by

$$Pb(t)=\exp(Mt)Pb(0)+\int_0^t ds\,\exp[M(t-s)]Pg , \tag{A4}$$

where one has ignored the time dependence of g through x_i providing that $\dot{x}_i$ and $\ddot{x}_i$ are sufficiently small when two pulses are distant apart. Noting the relation (A3), the solution (A4) can be written as

$$Pb(t)=Pb(0)+t\{MPb(0)+Pg\}+\tfrac{1}{2}t^2MPg . \tag{A5}$$

One imposes the requirement that the terms increasing with t should vanish. The second term in (A5) is unnecessary since it simply determines the initial value $b(0)$. The absence of the third term in (A5) leads us to

$$MPg=PMg=0 . \tag{A6}$$

Using (A1) with $\varphi=g$, one obtains from (A6)

$$(g,\Psi)=0 , \tag{A7}$$

where one has used the relation (A2). Thus, the orthogonality condition (A7) eliminates the secular part in (A5).

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